B. Math. (Hons.) Second Year
Second Midsemestral Examination 2024-25
Ring Theory
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Q 1. (10)
Indicate if the following is true or false; give a single brief sentence of reasoning if true, and a counter example, if false:
(a) $\mathbb{Z}_{2025}$ is a ring that contains some non-zero nilpotent elements.
(b) In any integral domain, all non-zero prime ideals are maximal,
(c) In the ring $M_2(\mathbb{R})$, if an element has a left inverse, then that left inverse must also be a right inverse.
(d) If S is a multiplicative set in a commutative ring A with unity, then the complement of S must be a union of prime ideals.
(e) For a given ring R, another ring S is $\cong R/I$ for some two-sided ideal I of R if, and only if, there exists a ring homomorphism from R ONTO S,
(f) If $T : A \to \mathbb{Z}_p$ is an 1-1 homomorphism, where A is a commutative ring with unity, and p is prime, then the only idempotents of A are 0 and 1.
(g) In a commutative ring A with unity, the set of nilpotent elements is necessarily contained in the Jacobson radical of A.
(h) The only field of characteristic 2 is $\mathbb{Z}_2$.
(i) The direct product $R_1 \times R_2$ of two non-zero rings is never an integral domain.
(j) In a finite commutative ring R with unity, every prime ideal is maximal.

Q 2. Consider the group ring $\mathbb{C}[S_n]$. If $D_1, D_2, \cdots D_{p(n)}$ are the conjugacy classes of S_n, prove that the elements $a_i = \sum_{\sigma \in D_i} \sigma$ (for $i = 1, \cdots, p(n)$) are in the center of the group ring.

OR

Q 2. Let R be a commutative ring with unity 1. If I, J are ideals, prove that

$$R \to R/I \times R/J;$$

$$r \mapsto (r + I, r + J)$$

is surjective if, and only if, $I + J = R$.

Q 3. (3+7)
(i)Prove that the set of nilpotent elements in a noncommutative ring may not form an additive subgroup.
(ii) Show that a commutative ring R which has no non-zero nilpotent elements is isomorphic to a subring of a direct product of integral domains.
Hint for (ii). Prove (and use) that the nilpotent elements form the intersection of all prime ideals.

OR

Q 3. (4+6)
Let $R := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Q}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$.
Prove:
(i) R is a subring consisting of all matrices in $M_2(\mathbb{Q})$ of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.
(ii) $R \cong \mathbb{Q}[X]/(X^2)$ as rings.

Q 4. Consider the ring homomorphism

$$\theta : \mathbf{C}[X,Y] \to \mathbf{C}[T];$$
$$X \mapsto T^2,\ Y \mapsto T^3.$$

Prove that Ker $\theta = (X^3 - Y^2)$.

OR

Q 4. Prove that $\mathbb{Z}[i]/(1+ni) \cong \mathbb{Z}/(1+n^2)\mathbb{Z}$ as rings, where n is any positive integer.
Hint. No number theory is needed.

Q 5. Let A be a commutative ring with unity, let P be an ideal, and let $a \in A$. If $I + (a)$ and $I : (a)$ are finitely generated ideals, prove that I must be finitely generated as well. Deduce that if all prime ideals are finitely generated, then A must be Noetherian.

OR

Q 5. Let A be a commutative ring with unity. Let $P_1, \cdots, P_n$ be prime ideals such that $\bigcup_{i=1}^n P_i \supset I$ for an ideal I. Prove that $P_i \supset I$ for some i.

Q 6. Let A be a commutative ring with unity, which is Noetherian. If $T : A \to A$ is a surjective ring homomorphism, prove that T must be an isomorphism.
Hint. Consider an increasing sequence of ideals constructed using T.

OR

Q 6. Show that $\mathbb{C}[X,Y]/(X^2 + Y^2 - 1) \cong \mathbb{C}[T, T^{-1}]$. Hence, deduce that every ideal is principal.